Dirac's Relativistic Equation

In 1962, Dirac formulated an equation to avoid the difficulties arising in Klein Gordan equation. Lorentz invariance demand that an equation which is linear in H and hence E must also be linear in P. This is because both E & P enter linearly in four momentum pll given by $(P, i\frac{E}{c})$

Dirac approached the problem of finding a relativistic wave equation form.

$$\hat{H}\psi(r,t) = i\hbar \frac{\partial \psi(r,t)}{\partial t} \quad \text{— ①}$$

Linearization of Hamiltonian H → The simplest linearized Hamiltonian for a free particle.

$$H = c\vec{\alpha}\cdot P + \beta mc^2 \quad \text{— ②}$$

Substituting H from ② in ①

$$(c\alpha\cdot\vec{P} + \beta mc^2)\psi(r,t) = i\hbar \frac{\partial \psi(r,t)}{\partial t} \quad \text{— ③}$$

$$[c\vec{\alpha}\cdot(-i\hbar\nabla) + \beta mc^2]\psi(r,t) = i\hbar\frac{\partial\psi(r,t)}{\partial t}$$

$$\left(i\hbar\frac{\partial}{\partial t} + i\hbar\vec{\alpha}\cdot\nabla - \beta mc^2\right)\psi(r,t) = 0 \quad \text{— ④}$$

As already indicated if this equation is to describe a free particle, there can be no term in the Hamiltonian that depend upon the space and time coordinates. Consequently α and β are independent of r, t, p and E and hence commute with all of them.

For simplicity writing again E for $i\hbar \frac{\partial}{\partial t}$ and p for $-i\hbar \nabla$.

$$(E - c\vec{\alpha}\cdot p - \beta mc^2)\psi(r,t) = 0 \quad - (5)$$

operating above equation by $[E + c\vec{\alpha}\cdot p + \beta mc^2)$

$$(E + c\vec{\alpha}\cdot p + \beta mc^2)(E - c\vec{\alpha}\cdot p - \beta mc^2)\psi = 0$$

$$[E^2 - (c\alpha\cdot\vec{p} + \beta mc^2)^2]\psi = 0$$

$$[E^2 - c^2(\vec{\alpha}\cdot p)^2 - \beta^2 m^2 c^4 - mc^3(\vec{\alpha}\cdot p) - mc^3 \beta\vec{\alpha}\cdot p]\psi = 0 \quad (6)$$

$$\vec{\alpha} = i\alpha_x + j\alpha_y + k\alpha_z$$

$$\vec{p} = iP_x + jP_y + kP_z$$

$$\vec{\alpha}\cdot\vec{p} = \alpha_x P_x + \alpha_y P_y + \alpha_z P_z$$

Therefore equation (6) becomes.

$$[E^2 - c^2(\alpha_x P_x + \alpha_y P_y + \alpha_z P_z)^2 - \beta^2 m^2 c^4 - mc^3(\alpha_x P_x + \alpha_y P_y + \alpha_z P_z)\beta$$
$$- mc^3 \beta(\alpha_x P_x + \alpha_y P_y + \alpha_z P_z)]\psi = 0$$

or $$[E^2 - c^2\{\alpha_x^2 P_x^2 + \alpha_y^2 P_y^2 + \alpha_z^2 P_z^2 + (\alpha_x\alpha_y + \alpha_y\alpha_x)P_x P_y + (\alpha_y\alpha_z + \alpha_z\alpha_y)P_y P_z$$
$$+ (\alpha_z\alpha_x + \alpha_x\alpha_z)P_z P_x\} - \beta^2 m^2 c^4 - mc^3\{(\alpha_x\beta + \beta\alpha_x)P_x + (\alpha_y\beta + \beta\alpha_y)P_y$$
$$+ (\alpha_z\beta + \beta\alpha_z)P_z]\psi = 0 \quad — (8)$$

Where the substitution $i\hbar\frac{\partial}{\partial t}$ for E and $i\hbar\nabla$ for P are implied K.G. equ^n is

$$[E^2 - c^2(P_x^2 + P_y^2 + P_z^2) - m^2c^4]\psi = 0 \quad — (9)$$

Comparing equation 8 and 9

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$$
$$\alpha_x\alpha_y + \alpha_y\alpha_z = 0, \; (\alpha_y\alpha_z + \alpha_z\alpha_y) = 0, \; (\alpha_z\alpha_x + \alpha_x\alpha_z) = 0$$
$$\alpha_x\beta + \beta\alpha_x = 0, \; (\alpha_y\beta + \beta\alpha_y) = 0 \; (\alpha_z\beta + \beta\alpha_z) = 0$$

Four quantities $\alpha_x, \alpha_y, \alpha_z$ and β have the following properties.
(i) Their square are unity (ii) and they anti commute with one another in pair
~~qant~~ quantities like these can be expressed in term of matrix.